OPTIMUM DESPINNING (BRAKING) REGIMES FOR THE ROTATIONAL MOTION OF A SYMMETRIC BODY

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Control of the rotating motion of a rigid body by means of torque producing rockets is reduced to the selection of thrust programing regimes for the rockets in conjunction with the conditions for a specific problem. Below is investigated the problem of determining the optimum law for rocket operation during despinning of the angular velocity of a symmetric body which initially rotates freely in space about its center of mass. Two despinning regimes are considered: despinning in shortest time (for unspecified fuel consumption), and despinning with minimal fuel consumption (for unspecified time). Although both of these regimes can coincide in certain specific cases, they are different in general and must therefore be considered separately.

It is assumed in the present analysis that the rockets produce control moments about the principal axes of inertia of the body and that the moments of inertias of the body (as well as the directions of the principal body axes) remain practically unchanged as the result of fuel consumption. The control moments are considered bounded in magnitude.

It was found, as the result of the investigation, that in the first case all three control moments act reversely until the body comes to a rest, and in the second case the transverse moments are activated sequentially while the longitudinal moment (directed along the symmetry axis of the body) remains on until complete elimination of the longitudinal component of the body angular velocity. Phase trajectories are presented for the problem of precessional motion elimination in a body whose longitudinal velocity components remains unchanged.

1. The despinning in shortest time. Let us consider the problem of finding the optimum regime with respect to speed for slowing down the motion of a symmetric rigid body. Assuming for definiteness that the polar moment of inertia C of the body is larger than its equatorial moment of inertia A, the system of differential equations of the motion of the body may be expressed as

$$\omega_x^{\cdot} + \varepsilon\omega_y\omega_z = \frac{m_x}{A}, \qquad \omega_y^{\cdot} - \varepsilon\omega_x\omega_z = \frac{m_y}{A}, \qquad \omega_z^{\cdot} = \frac{m_z}{C}$$

$$\left(\varepsilon = \frac{C-A}{A} > 0\right) \tag{1.1}$$

It is required to determine the control law for the torques m_x, m_y, m_z

(with respect to the principal axes of inertia x, y, z) such that the components of the angular velocity ω_x, ω_y, ω_z (which become phase coordinates in the sequel) attain specified final values in shortest time. In case of a complete stop, the final values of the body's angular velocities must become zero, while in the case of incomplete despinning, for example, when only precessional motion is removed, only the transverse components $\omega_x(T)$ and $\omega_y(T)$ must vanish, where T is final time of the process. In view of the linearity of the control moments with respect to the derivatives in Equations (1.1), it is convenient to utilize the maximum principle [1] for formulating the variational problem. Let us construct the H function for the problem considered as

$$H = \sum_k p_k \omega_k = p_x \left(\frac{m_x}{A} - \varepsilon\omega_y\omega_z\right) + p_y \left(\frac{m_y}{A} + \varepsilon\omega_x\omega_z\right) + p_z \frac{m_z}{C} \qquad (1.2)$$

and write down the system of equations for phase impulses as

$$p_k^{\cdot} = -\, \partial H / \partial \omega_k$$

Expanded, it becomes (1.3)

$$p_x^{\cdot} + \varepsilon p_y \omega_z = 0, \qquad p_y^{\cdot} - \varepsilon p_x \omega_z = 0, \qquad p_z^{\cdot} - \varepsilon\,(p_x\omega_y - p_y\omega_x) = 0$$

Without yet fixing the boundary conditions for the problem, we will construct the necessary integrals of Equations (1.1) and (1.3) and will investigate the general character of the optimum control regime. Utilizing the maximum principle, we will establish the optimum law for variation of the controls m_k. Since the function H is linearly dependent on the controls, it becomes maximum for limiting values of the controls, and if the multiplier p_k for m_k is positive then the control is at its upper bound, and conversely. Thus, for $p_k \not\equiv 0$ the optimum regime for the moment m_k variation will be of relay (*) type and is determined by the following relationships:

$$m_k(t) = \max m_k \quad \text{for } p_k(t) > 0, \qquad m_k(t) = \min m_k \quad \text{for } p_k(t) < 0 \qquad (1.4)$$

If the variation regions of m_k are symmetric with respect to zero, then

$$m_k(t) = \max|m_k| \operatorname{sgn} p_k(t) \qquad (1.5)$$

Here $\max|m_k|$ is the magnitude of the kth control amplitude. In the following, m_k will denote in the present problem the quantity defined by the relation (1.5).

2. Integration of the equations for optimum motion. In order to obtain the solution of the system of equations (1.1) and (1.3) in which the controls are defined in accordance with the conditions (1.5), we will separate at first the following subsystem containing the variables p_x, p_y and ω_z

$$p_x^{\cdot} + \varepsilon p_y \omega_z = 0, \qquad p_y^{\cdot} - \varepsilon p_x \omega_z = 0, \qquad \omega_z^{\cdot} = m_z / C \qquad (2.1)$$

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*) If $p_k \equiv 0$, then singular regimes occur which may differ from the relay type. Lee [3] considers these regimes in greater detail.

In accordance with (1.5), the moment m_z is constant on each separate segment of the motion, and consequently, the solution of the last equation on the segment will be

$$\omega_z = \omega_{z0} + C^{-1} m_z t \tag{2.2}$$

Thus, the zth component of the body's angular velocity in the optimum despinning regime represents a piece-wise linear function the discontinuity points of which, according to (1.5), correspond to the roots of the function p_z.

Let us consider the complex function

$$p = p_x + i p_y \tag{2.3}$$

The first two equations in (2.1) yield

$$p^{\cdot} - i\varepsilon\omega_z p = 0 \tag{2.4}$$

The solution of this equation is

$$p = p_0 \exp \left(i\varepsilon \int_0^t \omega_z \, dt\right) \tag{2.5}$$

Taking into account Expression (2.2), we find

$$\varepsilon \int_0^t \omega_z \, dt = \frac{\varepsilon m_z}{2C} t^2 + \varepsilon \omega_{z0} t = \lambda (\omega_z^2 - \omega_{z0}^2) \qquad \left(\lambda = \frac{\varepsilon C}{2 m_z}\right) \tag{2.6}$$

Now, in place of (2.5), we get

$$p \exp (-i\lambda\omega_z^2) = \text{const} \tag{2.7}$$

The expression for the function p may also be given as

$$p = P \exp [i (\lambda\omega_z^2 + \alpha)] \tag{2.8}$$

where P and α are real constants of integration determined from the boundary conditions of the problem. Therefore, we have for p_x and p_y, respectively

$$p_x = P \cos (\lambda\omega_z^2 + \alpha), \qquad p_y = P \sin (\lambda\omega_z^2 + \alpha) \tag{2.9}$$

Multiplying the second equation in (1.1) by i and adding it to the first one, we obtain the equation for the complex transverse angular velocity of the body

$$\omega^{\cdot} - i\varepsilon\omega_z\omega = A^{-1} m \qquad (\omega = \omega_x + i\omega_y, \ m = m_x + i m_y) \tag{2.10}$$

The general integral of this equation is according to [2] of the form

$$\omega = \left[\omega_0 + \frac{m}{A} \int_0^t \exp\left(-i\varepsilon \int_0^t \omega_z dt\right) dt\right] \exp\left(i\varepsilon \int_0^t \omega_z dt\right) \tag{2.11}$$

Taking into account Expression (2.6) and changing from the variable t to ω_z, we obtain as the result of integration

$$\omega = \omega_0 \exp [i\lambda (\omega_z^2 - \omega_{z0}^2)] + \frac{m (\varepsilon + 1)}{m_z} \left(\frac{\pi}{2|\lambda|}\right)^{1/2} [C - C_0 - \\ - i \operatorname{sgn} \lambda (S - S_0)] \exp (i\lambda\omega_z^2) \tag{2.12}$$

$$(S = S (\omega_z \sqrt{|\lambda|}), \quad S_0 = S (\omega_{z0} \sqrt{|\lambda|}))$$

Here $C(w)$ and $S(w)$ are the Fresnel integrals

$$C(w) = \sqrt{\frac{2}{\pi}} \int_0^w \cos v^2 dv, \qquad S(w) = \sqrt{\frac{2}{\pi}} \int_0^w \sin v^2 dv \tag{2.13}$$

The integral (2.12) can also be expressed as

$$\omega \exp(-i\lambda\omega_z^2) - \frac{m(\varepsilon+1)}{m_z} \sqrt{\frac{\pi}{2|\lambda|}} [C - i \operatorname{sgn} \lambda S] = D = \text{const} \tag{2.14}$$

To determine function p_z, one need not integrate the corresponding differential equation; this function is part of the problem integral H which may be expressed in the form

$$m_z C^{-1} p_z - \varepsilon\omega_z (p_x\omega_y - p_y\omega_x) + A^{-1}(p_x m_x + p_y m_y) = 1 \tag{2.15}$$

The right-hand part of this integral only is constant during the entire process of despinning. This is because here the relay controls m_k are the multipliers of the corresponding p_k functions, which vanish at switching instants.

3. Investigation of the solution. The derived integrals for phase coordinates and impulses contain a sufficient number of constants for the solution of the considered two-point boundary value problem. The three constants defining the initial position of the phase point in phase space can be chosen as, for example, the initial values of the velocities ω_{x0}, ω_{y0}, ω_{z0}. Then, in order to satisfy the boundary conditions of the problem (for example, to get $\omega_x(T) = \omega_y(T) = \omega_z(T) = 0$, in the case of a complete stop) there remain the quantities P, α and T, where T is the duration of the despinning process, and P and α are integration constants in Equations (2.9). From the requirement of continuity for p_x and p_y at the switching points for the torques m_k it follows that their amplitude P and phase $(\lambda\omega_z^2 + \alpha)$ also must be continuous. This means in turn that the constant P retains its value during the entire despinning process, and the constant α changes stepwise by a quantity $\pm 2|\lambda|\omega_z^2$ at the switching points of the torque m_z, remaining unchanged at the switching points for the torques m_x and m_y.

Taking into account Expressions (2.9) for p_x and p_y and also considering the relations (1.5), we rewrite the integral (2.15) in the form

$$\begin{aligned} |p_z m_z| C^{-1} + PA^{-1} [|m_x \cos(\lambda\omega_z^2 + \alpha)| + |m_y \sin(\lambda\omega_z^2 + \alpha)|] - \\ - \varepsilon P\omega_z [\omega_y \cos(\lambda\omega_z^2 + \alpha) - \omega_x \sin(\lambda\omega_z^2 + \alpha)] = 1 \end{aligned} \tag{3.1}$$

Hence, it follows that the derived integral determines only the modulus of the function p_z and not its sign. The $|p_z|$ can also be expressed as an explicit function of ω_z, having eliminated ω_x and ω_y by means of the expression (2.14). However, for computing purposes, it is apparently more convenient to use Expression (3.1) directly.

The laws for switching the torques m_x and m_y are given by Expressions

$$m_x = |m_x|_{\max} \operatorname{sgn} \cos(\lambda\omega_z^2 + \alpha), \qquad m_y = |m_y|_{\max} \operatorname{sgn} \sin(\lambda\omega_z^2 + \alpha) \qquad (3.2)$$

At the same time, the switching of the torques m_x and m_y occurs, respectively, at the points defined by Equations

$$\lambda\omega_z^2 + \alpha = n\pi + {}^1/_2\pi, \qquad \lambda\omega_z^2 + \alpha = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots) \qquad (3.3)$$

Phase trajectories of the system are determined by Equation (2.14), the right-hand side of which changes stepwise at the switching points of the torques m_x, m_y, m_z. Let us find the magnitude of this step at the switching point of m_z. On the strength of the continuity of ω and ω_z we obtain

$$\Delta D\ (\Delta m_z) = D_2 - D_1 = 2i\omega \sin \lambda\omega_z^2 + \frac{2m\,(\varepsilon+1)}{(m_z)_1}\left(\frac{\pi}{2\,|\,\lambda\,|}\right)^{1/2} \mathrm{C} \qquad (3.4)$$

At the switching point of the torque m we get for ΔD

$$\Delta D\ (\Delta m) = D_2 - D_1 = \frac{(m_1 - m_2)\,(\varepsilon+1)}{m_z}\left(\frac{\pi}{2\,|\,\lambda\,|}\right)^{1/2} (\mathrm{C} - i \operatorname{sgn} \lambda\, \mathrm{S}) \qquad (3.5).$$

Here the subscript 1 denotes a quantity before switching and subscript 2 that after switching, D is a constant defined by the integral (2.14). The relationships (3.4) and (3.5) permit joining of the phase trajectory of the switching points of the controls during numerical evaluation of the despinning process. Equation (2.14) is equivalent to the two real equations

$$\omega_x \cos \lambda\omega_z^2 + \omega_y \sin \lambda\omega_z^2 = \operatorname{Re} D + \frac{\varepsilon+1}{m_z}\left(\frac{\pi}{2\,|\,\lambda\,|}\right)^{1/2} (m_x\, \mathrm{C} + m_y \operatorname{sgn} \lambda\, \mathrm{S})$$

$$\omega_x^2 + \omega_y^2 = |\,D\,|^2 + \frac{2\,(\varepsilon+1)}{m_z}\left(\frac{\pi}{2\,|\,\lambda\,|}\right)^{1/2} [\mathrm{C} \operatorname{Re}(D\bar{m}) - \operatorname{sgn} \lambda\, \mathrm{S} \operatorname{Im}(D\bar{m})] +$$
$$+ \frac{|\,m\,|^2\,(\varepsilon+1)^2\,\pi}{2m_z^2\,|\,\lambda\,|}(\mathrm{C}^2 + \mathrm{S}^2) \qquad (3.6)$$

This shows that the phase trajectory represents a line of intersection of a ruled helicoidal surface with a surface of revolution, i.e. a certain space spiral-like curve of varying radius and varying pitch which turns itself about the ω_z-axis.

For sufficiently large values of ω_z when $\lambda\omega_z^2 \gg 1$, the expression for the angular velocity ω as a function of ω_z can be simplified. Substituting asymptotic expansions for the Fresnel integrals

$$\mathrm{C}\,(\omega_z \sqrt{|\,\lambda\,|}) \approx \frac{1}{2} + \frac{\sin |\,\lambda\,|\,\omega_z^2}{\omega_z \sqrt{2\pi\,|\,\lambda\,|}}, \qquad \mathrm{S}\,(\omega_z \sqrt{|\,\lambda\,|}) \approx \frac{1}{2} - \frac{\cos |\,\lambda\,|\,\omega_z^2}{\omega_z \sqrt{2\pi\,|\,\lambda\,|}} \qquad (3.7)$$

into (2.14) and neglecting nonreal constants, we get

$$\left(\omega - \frac{im}{\omega_z\,(C - A)}\right) \exp\,(-\,i\lambda\omega_z^2) = \mathrm{const} \qquad (3.8)$$

Multiplying both parts of (3.8) by their conjugate quantities we find

$$\left(\omega_x + \frac{m_y}{\omega_z\,(C - A)}\right)^2 + \left(\omega_y - \frac{m_x}{\omega_z\,(C - A)}\right)^2 = \mathrm{const} \qquad (3.9)$$

As can be seen from above, for sufficiently large values of ω_z, the projection of the phase trajectory on the surface $\omega_x\omega_y$ will consist in a sequence of arcs similar to the arcs of circles.

Thus, the construction of the solutions for the coordinates ω_k and the impulses p_k permit in principle the evaluation of a nonsingular optimum regime for despinning of a symmetric body. However, these integrals contain two unknown constants P and α determined from the boundary conditions of the problem. Therefore, for specific evaluation of the despinning process, it is more convenient to consider it in the reverse direction assuming, for example, that initially $\omega_x = \omega_y = \omega_z = 0$. Then assuming values for P and α , it is possible to compute the entire phase trajectory up to a certain point $\omega_x(T)$, $\omega_y(T)$, $\omega_z(T)$. Varying then the initial values of P and α it is possible to "fall" into a point of phase space separated by an arbitrarily small distance from the given point. We will note that the formula for the initial value of the modulus p_{z0}

$$|p_{z0}| = \frac{C}{|m_z|}\Big[1 - \frac{P}{A}\left(|m_x \cos\alpha_0| + |m_y \sin\alpha_0|\right)\Big] \tag{3.10}$$

yields the condition

$$0 \leqslant P < \frac{A}{|m_x \cos\alpha_0| + |m_y \sin\alpha_0|} \tag{3.11}$$

Depending on the choice of the sign for the quantity p_{z0} we get two phase trajectories which are symmetric with respect to the surface $\omega_z = 0$. For definiteness, it can be assumed that $p_{z0} > 0$ and, consequently, $m_{z0} > 0$ and $\lambda_0 > 0$. The signs of the control torques m_{x0} and m_{y0} ,depending on the magnitude of the angle α_0, are determined with the aid of Expressions (3.2).

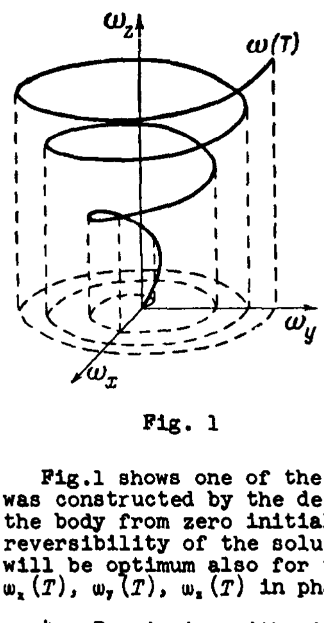

Fig. 1

Fig.1 shows one of the phase trajectories for the considered problem which was constructed by the described method, i.e. for the spinning-up process of the body from zero initial velocity to its final value. By virtue of the reversibility of the solution for the variational problem, this trajectory will be optimum also for the despinning process of the body from the point $\omega_x(T)$, $\omega_y(T)$, $\omega_z(T)$ in phase space to the point $\omega_x = \omega_y = \omega_z = 0$.

4. Despinning with minimal consumption of mass. Assuming that the control torques m_k are generated by rockets with constant exhaust velocities, and also that the rockets producing m_x and m_y torques are identical, we take the following integral as the functional of the problem:

$$M = \int_0^T \left(\mu|m_x| + \mu|m_y| + \nu|m_z|\right) dt \tag{4.1}$$

where M is the quantity proportional to the consumed mass for despinning and μ and ν are positive constants. Thus, in this problem, the system of equations of motion (1.1) should be augmented by the additional equation for the function M

$$M^{\cdot} = \mu|m_x| + \mu|m_y| + \nu|m_z| \tag{4.2}$$

and we should seek such a switching regime for the control torques m_k for which $M(T)$ shall be minimal. However, Equation (4.2) is awkward in that it contains absolute values of sign varying functions. In order to eliminate this inconvenience, we will assume that the quantities m_k can be only positive or equal to zero

$$0 \leqslant m_k \leqslant |m_k|_{\max} \tag{4.3}$$

and their sign in Equations (1.1) will be given by introducing additional controls u_k, the complete set of permissible values of which is bounded only by two points $u_k = \pm 1$. Then in place of the systems (1.1) and (4.2) we will have

$$\omega_x^{\cdot} + \varepsilon\omega_y\omega_z = \frac{m_x u_x}{A}, \quad \omega_y^{\cdot} - \varepsilon\omega_x\omega_z = \frac{m_y u_y}{A}$$

$$\omega_z^{\cdot} = \frac{m_z u_z}{C}, \quad M^{\cdot} = \mu m_x + \mu m_y + \nu m_z \tag{4.4}$$

It is easy to see that the differential equations and, consequently, the integrals for p_x and p_y will remain as before. The integral which determines p_z, however, will change because the form of the function H will change with the introduction therein of a new quantity p_M, which is conjugate of the variable M. It follows from the formulation of the problem that the quantity p_M will be the only constant equal to unity during the entire despinning process. With this in mind, the function H can be expressed in the form

$$H = m_x\left(\frac{p_x u_x}{A} - \mu\right) + m_y\left(\frac{p_y u_y}{A} - \mu\right) +$$

$$+ m_z\left(\frac{p_z u_z}{C} - \nu\right) - \varepsilon\omega_z(p_x\omega_y - p_y\omega_x) \tag{4.5}$$

From the maximum condition for this function with respect to the controls m_k and u_k, it follows that

$$u_k = \operatorname{sgn} p_k, \qquad m_k = |m_k|_{\max} \operatorname{Sg}(p_k u_k - I_k) \tag{4.6}$$

Here

$$I_x = I_y = \mu A, \quad I_z = \nu C, \quad \operatorname{Sg} w = 1 \text{ for } w > 0, \quad \operatorname{Sg} w = 0 \text{ for } w < 0$$

As far as the function p_z is concerned, it is determined directly from the integral (4.5), as before, which in view of (4.6) can be expressed as follows:

$$m_x\left(\frac{|p_x|}{A} - \mu\right) + m_y\left(\frac{|p_y|}{A} - \mu\right) + m_z\left(\frac{|p_z|}{C} - \nu\right) = \omega_z p_z^{\cdot} \tag{4.7}$$

Since the left-hand part of this equation cannot be negative in the optimum regime, then

$$\operatorname{sgn} p_z^{\cdot} = \operatorname{sgn} \omega_z \tag{4.8}$$

It follows from the relation (4.7) that at the end of the despinning process when $\omega_x = \omega_y = \omega_z = 0$, the left-hand part of it must also be equal to zero, i.e. conditions

$$|p_k(T)| \leqslant I_k \tag{4.9}$$

must be fulfilled.

In those cases when at some section the condition

$$|p_k| = I_k \tag{4.10}$$

is fulfilled there can arise singular control regimes when the magnitude of the control torques may assume other than its boundary values.

Apparently, the singular regime for transverse torques m_x and m_y can arise only simultaneously for both of them and only when the longitudinal velocity ω_z is entirely eliminated, since in the opposite case, according to (2.4), the functions p_x and p_y cannot become constant quantities fulfilling condition (4.10). The singular regime for the longitudinal torque m_z is characterized by conditions $\omega_z \neq 0$ and $|p_z| = \nu C$. It follows that $p_z^\cdot = 0$, i.e. either $\omega_x = \omega_y = 0$ or $p_x \omega_y = p_y \omega_x$.

The first case corresponds to the regime of pure rotation of the body about the longitudinal axis, and here, naturally, the law for variation of m_z does not affect the general consumption of fuel.

As can be easily shown, the second case can take place only in passive zones of the transverse torque, i.e. on sections where $m = 0$. Note that in general the derivative $p_z^\cdot$ becomes constant in passive zones where $m = 0$ (as can be easily shown with the aid of Expressions (2.5) and (2.11) and, consequently, the function p_z is linear in these intervals of time.

It follows from the form of the integral (4.7) that during the entire despinning process the function p_z varies monotonously, continously increasing or decreasing depending on the sign of the initial velocity ω_{z0}. If it is assumed that initially $\omega_{z0} > 0$, then according to (4.8) $p_{z0}^\cdot > 0$ and p_z will increase. In this case, it is necessary that condition $p_{z0} < -\nu C$ be fulfilled, since otherwise the angular velocity ω will begin to increase continuously and no despinning will occur in view of Equations (4.4) and (4.6).

Analogously, if $\omega_{z0} < 0$, then the initial value of p_{z0} must be larger than νC. Since this reasoning is valid for any instant of time, it can be concluded that the sign of the function p_z must always be opposite to the sign of ω_z, i.e.

$$u_z = -\operatorname{sgn} \omega_z \tag{4.11}$$

Thus, the despinning of the longitudinal angular velocity occurs by turning on the torque m_z in the direction opposite to ω_z. If the magnitude of this torque remains constant at all times, then the duration of the action τ is defined by Formula

$$\tau = \frac{|\omega_{z0}|\, C}{|m_z|} \tag{4.12}$$

whereby

$$|p_z(\tau)| = \nu C \tag{4.13}$$

The law for variation of ω_z will then be linear so that, if for definiteness $\omega_{z0} > 0$, we get

$$\omega_z = \omega_{z0} - m_z C^{-1} t \tag{4.14}$$

In view of this, ω_z can be utilized instead of t as the independent variable, as long as $t < \tau$, i.e.before the velocity ω_z becomes zero. By the same token, the integrals (2.8) and (2.14) remain valid on the section $0 \leqslant t \leqslant \tau$ which only require the substitution of the controls u_k before the corresponding torques m_k . Also, it is understood, and should be noted that in contrast to the problem of speed control here the controls m_x, m_y have passive zones, i.e. sections where $m_k = 0$ $(k = x, y)$.

The location of these zones is defined by condition

$$|p_x| \leqslant \mu A, \qquad |p_y| \leqslant \mu A \tag{4.15}$$

From the requirement that these conditions must be fulfilled simultaneously in finite time, we obtain the inequality for the amplitude P

$$\mu A \leqslant P \leqslant \mu A \sqrt{2} \tag{4.16}$$

5. Despinning of the transverse velocity. A particular case of the considered problem is the problem of eliminating the precessional motion of the rigid body, i.e. the removal of the transverse components of angular velocity ω_x and ω_y of the symmetric body with $m_z \equiv 0$. Under these conditions, the form of the solution will be different since $\omega_z = \omega_{z0} = \text{const}$ and, consequently, instead of the system (1.1) we shall have a system of linear equations with constant coefficients. Denoting

$$\varepsilon\omega_{z0} = \rho \tag{5.1}$$

for the complex angular velocity ω and function $p = p_x + ip_y$ we will have the integrals

$$\left(\omega - \frac{im}{\rho A}\right) e^{-i\rho t} = \text{const}, \qquad pe^{-i\rho t} = \text{const} \tag{5.2}$$

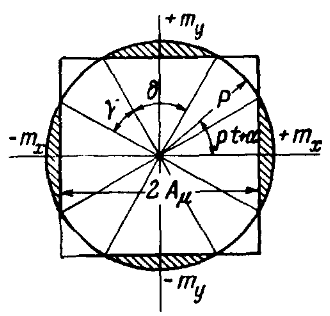

Fig. 2

It follows from this that the phase trajectory on the surface $\omega_x\omega_y$ will be a curve composed from circular arcs. In contrast to the problem of speed control, here the composition of these arcs will consist also of the arcs with centers at the origin of the coordinates corresponding to those zones where $m_x = m_y = 0$.

In order to construct the phase trajectory of the system, we first construct a diagram in the form of a circle of radius P and a square with a side $2\mu A$ as is shown in Fig. 2. In moving along this circle, the end of the vector $p = Pe^{i(\rho t+\alpha)}$ will then fall into zones corresponding to the various values of the controls m_x and m_y and, as can be seen, the shaded segments correspond to the torques m_x and m_y acting while the unshaded segments correspond to passive zones, i.e. when $m_x = m_y = 0$. It is easy to see that the width of the passive zone γ as well as the width of any active zone δ depends on the magnitude of the ratio $\mu A/P$ and are subject to the relationships

$$\cos {}^1\!/_2\delta = \mu A \,/\, P, \qquad \gamma + \delta = {}^1\!/_2\pi \tag{5.3}$$

With the aid of this diagram, we can construct a family of phase trajectories stemming from the origin of the coordinates. Apparently, the first part must be active, for example, $m_x \neq 0$ and for definiteness, we assume that $\cos \alpha_0 > \mu A/P$, so that $u_x = 1$.

Then, according to (5.2), the equation for the initial part of the phase curve will be

$$\omega_x^2 + \left(\omega_y - \frac{m_x}{\rho A}\right)^2 = \left(\frac{m_x}{\rho A}\right)^2 \tag{5.4}$$

As soon as the complex vector p in Fig. 2 reaches $\frac{1}{2}\delta$, i.e. the complex vector $\omega - i(m_x/\rho A)$ turns in the phase plane by an angle $\frac{1}{2}\delta - \alpha$ about the point a_1 , the torque m_x is turned off and then the phase curve becomes an arc of a circle centered about the origin of the coordinates with the enclosed angle γ. Then the torque m_y is turned on and the phase trajectory moves along the circle arc with the center at the point $a_2(\omega_y = 0, \omega_x = -m_y / \rho A)$, and the length of the arc will, apparently, enclose the angle δ. Thereafter, the torque m_y is turned off and the trajectory becomes the arc of the sector γ referred to the origin of the coordinates.

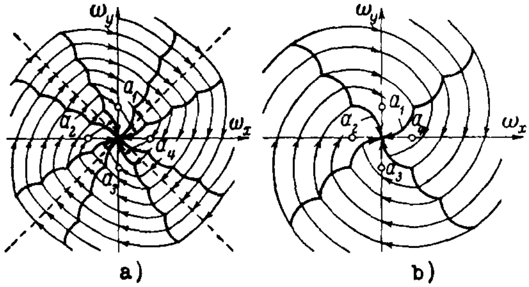

Fig. 3

Continuing this process, it can be shown that the family of phase trajectories with axes ω_x, ω_y will be of the form shown in Fig.3a. As can be seen the entire phase plane is divided into eight sectors by switching lines composed of circle arcs the length of each in angular measure is equal to δ. Four of these sectors correspond to the active zones of despinning and four others to the passive zones with coasting. Note, that the width of the passive zone γ in the present problem where there is no control of the longitudinal velocity, can be arbitrary in the region $0 \leqslant \gamma \leqslant \frac{1}{2}\pi$. The width of this zone affects only the duration of the despinning process but does not affect the fuel consumption. In the limit when $\gamma \to 0$, passive zones disappear and the phase plane becomes as shown in Fig.3b. It is worth noting that here the appearance of the phase trajectories does not actually differ from the phase trajectories in the analogous case of speed control with the exception of rotation of the whole phase plane by the angle $\frac{1}{4}\pi$.

Thus, comparing the regimes optimum with respect to speed control and fuel consumption, it can be concluded that they differ qualitatively in that the latter have control torques m_x and m_y acting not simultaneously by sequentially, and that the longitudinal torque m_z is not reversing in the despinning process, its sign is always opposite to the sign of the longitudinal component of the angular velocity ω_z.

In conclusion, it should be noted that the practical realization of the considered optimum regimes is most difficult in view of complex character of the switching surfaces located in the phase space of the system. However, the constructed solutions make it possible to find the limiting speed of control or the economy of control system performance and to evaluate from this point of view the quality of an arbitrarily selected nonoptimum regime of control.

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